

# Algebraic treatments of the problems of the spin-1/2 particles in the one and two-dimensional geometry: a systematic study

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We consider solutions of the  $2 \times 2$  matrix Hamiltonians of the physical systems within the context of the  $su(2)$  and  $su(1,1)$  Lie algebra. Our technique is relatively simple when compared with the others and treats those Hamiltonians which can be treated in a unified framework of the  $Sp(4, R)$  algebra. The systematic study presented here reproduces a number of earlier results in a natural way as well as leads to a novel findings. Possible generalizations of the method are also suggested.

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## INTRODUCTION

During the last decade a great deal of attention has been paid to examine different quantum optical models. Recently some algebraic techniques which improve both analytical and numerical solutions of the problems, have been suggested and developed for some quantum optical systems[1, 2, 3, 4, 5, 6, 7, 8, 9]. In general, the study of two level-systems, in a one and two-dimensional geometry, coupled to bosonic modes has been the subject of intense attention because of its extensive applicability in the various fields of physics[10, 11, 12, 13, 14, 15, 16, 17].

The most general form of the Hamiltonian of a two level system in two dimensional geometry can be expressed as

$$H = H_0 + \beta\sigma_0 + (\kappa_1 a + \kappa_2 a^+ + \kappa_3 b + \kappa_4 b^+) \sigma_+ + (\gamma_1 a + \gamma_2 a^+ + \gamma_3 b + \gamma_4 b^+) \sigma_- \quad (1)$$

where  $H_0 = \hbar\omega_1 a^+ a + \hbar\omega_2 b^+ b$ ,  $\sigma_0, \sigma_+$  and  $\sigma_-$  are usual Pauli matrices,  $a, b$  and  $a^+, b^+$  are bosonic annihilation and creation operators, respectively and  $\omega_i, \beta, \kappa_i$  and  $\gamma_i$  are physical constants. The Hamiltonian (1) includes various physical Hamiltonians depending on the choice of the parameters. For instance, when  $\omega_1 = \omega_2 = \omega$ ,  $\kappa_2 = \kappa_3 = \gamma_1 = \gamma_4 = 0$  and  $\kappa_1 = \kappa_4 = \gamma_2 = \gamma_3 = \kappa$ , the Hamiltonian,  $H$ , reduces to the  $E \otimes \varepsilon$  Jahn-Teller (JT) Hamiltonian[11, 18], when  $\omega_1 = \omega_2 = \omega$ ,  $\kappa_2 = \kappa_3 = \gamma_1 = \gamma_4 = 0$  and  $\kappa_1 = -\kappa_4 = \gamma_2 = -\gamma_3 = \kappa$ , the Hamiltonian,  $H$ , becomes the Hamiltonians of quantum dots including spin-orbit coupling[17, 19]. One can also obtain Jaynes-Cummings (JC) Hamiltonian[16], modified JC Hamiltonian[20] as well as many other interesting physical Hamiltonians by an appropriate choices of the parameters,  $\omega_i, \kappa_i$ , and  $\gamma_i$  in (1). There exist a relatively large number of different approaches for the solutions of the eigenvalue problems in the literature; however we present here a systematic and a unified treatment for the determination of the eigenvalues and eigenfunctions of (1), in the context of the  $su(2)$  and  $su(1,1)$  Lie algebra. Furthermore, we develop an algorithm and routines to implement an exact solution scheme for determining eigenvalues and eigenfunctions of the Hamiltonian  $H$ .

It is well-known that the Lie algebraic techniques are very powerful in describing many physical problems while improving both analytical and numerical solutions as well as understanding the nature of physical structures. In this paper we concentrate our attention to the solution of the Hamiltonian,  $H$ , by constructing the proper realization of the algebras  $su(2)$  and  $su(1,1)$ . In a straightforward way, furthermore, we shall see that, the Hamiltonian (1) automatically leads to  $su(2)$  or  $su(1,1)$  algebras depending on the choices of the parameters  $\kappa_i$ , and  $\gamma_i$ . We also note that the algebras  $su(2)$  and  $su(1,1)$  describing symmetry of the Hamiltonian,  $H$ , can be imbedded into a larger algebra that contains both[21, 22]. This algebra is  $Sp(4, R)$  that provides a unified treatment of the various approaches to the solution of the such problems.

The paper is organized as follows. In section II we discuss the bosonisation of the physical Hamiltonians whose original forms are given as differential operators. In section III we briefly review the properties of  $su(2)$  and  $su(1,1)$  Lie algebras and introduce their bosonic realizations which we need to solve the various Hamiltonians. Our main procedure are presented in section IV, where we deal with the solution of the Hamiltonian (1). The results are contained in section V. Finally we conclude our results in section VI.

## REALIZATION OF THE PHYSICAL PROBLEMS BY BOSONS

One way to relate a Hamiltonian with an appropriate Lie algebra is to construct its bosonic and fermionic representation. We are interested in the two-level system in a one and two-dimensional geometry, whose Hamiltonians are given in terms of bosons-fermions or matrix-differential equations. Therefore, it is worth to express a suitable differential realizations of the bosons. By the use of the differential realization of the operators one can easily find the connection between boson-fermion and matrix differential equation formalism of the Hamiltonians. To this end, let us start by introducing the following differential realizations of the boson operators:

$$\begin{aligned} a^+ &= \frac{\ell}{2}(x + iy) - \frac{1}{2\ell}(\partial_x + i\partial_y), \\ a &= \frac{\ell}{2}(x - iy) + \frac{1}{2\ell}(\partial_x - i\partial_y), \\ b^+ &= \frac{\ell}{2}(x - iy) - \frac{1}{2\ell}(\partial_x - i\partial_y), \\ b &= \frac{\ell}{2}(x + iy) + \frac{1}{2\ell}(\partial_x + i\partial_y) \end{aligned} \quad (2)$$

where  $\ell = \sqrt{\frac{m\omega}{\hbar}}$  is the length parameter and the boson operators obey the usual commutation relations

$$[a, a^+] = [b, b^+] = 1; \quad [a, b^+] = [b, a^+] = [a, b] = [a^+, b^+] = 0. \quad (3)$$

In principle, if a Hamiltonian is expressed by boson operators, one could rely directly on the known formulae of the action of boson operators on a state with a defined number of particles without solving differential equations. Apart from the mentioned method, sometimes the Hamiltonians can be put in a simple form by using the transformation properties of the bosons. Now, we briefly discuss the bosonic construction of the various Hamiltonians.

### Hamiltonians of quantum dots including spin-orbit coupling

The origin of the Rashba spin-orbit coupling in quantum dots due to the lack of inversion symmetry which causes a local electric field perpendicular to the plane of heterostructure. In literature the Hamiltonian has been formalized in the coordinate-momentum space, leading to a matrix differential equation. The Hamiltonian representing the Rashba spin orbit coupling for an electron in a quantum dot can be expressed as[17]

$$H_R = \frac{\lambda_R}{\hbar} (p_y \sigma_x - p_x \sigma_y) \quad (4)$$

where  $\lambda_R$  represents the strength of the spin orbit coupling, which can be adjusted by changing the asymmetry of the quantum well via external electric field. Here the matrices  $\sigma_x$ , and  $\sigma_y$  are Pauli matrices. We assume that the electron is confined in a parabolic potential

$$V = \frac{1}{2}m^*\omega_0^2(x^2 + y^2) \quad (5)$$

here  $m^*$  is the effective mass of the electron and  $\omega_0$  is the confining potential frequency. The Hamiltonian describing an electron in two-dimensional quantum dot takes the form

$$H = \frac{1}{2m^*} (P_x^2 + P_y^2) + \frac{1}{2}g\mu B\sigma_0 + V + H_R. \quad (6)$$

The term  $\frac{1}{2}g\mu B\sigma_z$  introduces the Zeeman splitting between the (+)x-polarized spin up and (-)x-polarized spin down. The factors  $g$  is gyromagnetic ratio  $\mu$  is the Bohr magneton. The canonical momentum  $\mathbf{P} = \mathbf{p} + e\mathbf{A}$  is expressed in terms of the mechanical momentum  $\mathbf{p} = -i\hbar(\partial_x, \partial_y, 0)$  and the vector potential  $\mathbf{A}$  can be related to the magnetic field  $\mathbf{B} = \nabla \times \mathbf{A}$ . The choice of symmetric gauge vector potential  $\mathbf{A} = B/2(-y, x, 0)$ , leads to the following Hamiltonian

$$\begin{aligned} H_{dot} &= -\frac{\hbar^2}{2m^*} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2}m^*\omega^2(x^2 + y^2) + \\ &\quad \frac{1}{2}i\hbar\omega_c \left( y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \right) + \frac{1}{2}g\mu B\sigma_0 + H_R \end{aligned} \quad (7)$$

where  $\omega_c = eB/m^*$ , stands for the cyclotron frequency of the electron,  $\omega = \sqrt{\omega_0^2 + (\frac{\omega_c}{2})^2}$  is the effective frequency. The Hamiltonian  $H_{dot}$  describing a two-level fermionic subsystem coupled to two boson modes can be expressed as:

$$H = \hbar\omega(a^+a + b^+b + 1) + \frac{\hbar\omega_c}{2}(a^+a - b^+b) - \sqrt{\frac{m\omega}{4\hbar}}\lambda_R[(b^+ - a)\sigma_+ + (b - a^+)\sigma_-] + \frac{1}{2}g\mu B\sigma_0 \quad (8)$$

The Pauli matrices are given by

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad \sigma_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad (9)$$

It is worth to point out here that the success of our construction leads to the connection between  $H_{dot}$  and  $H$ .

### $E \times \varepsilon$ Jahn-Teller Hamiltonian

The  $E \times \varepsilon$  JT Hamiltonian describing a two level fermionic subsystem coupled to two boson modes can be expressed[11] as

$$H_{JT} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2}m^*\omega_0^2(x^2 + y^2) + \frac{\mu}{2}\sigma_0 + \kappa[(x + iy)\sigma_+ + (x - iy)\sigma_-]. \quad (10)$$

where  $\mu$  is the level separation and  $\kappa$  is the coupling constants. In terms of bosonic operators we can easily obtain

$$H_{JT} = \hbar\omega(a^+a + b^+b) + \frac{\mu}{2}\sigma_0 + \sqrt{\frac{m\omega}{4\hbar}}\kappa[(a + b^+)\sigma_+ + (a^+ + b)\sigma_-]. \quad (11)$$

The JT problem is an old one and the complete description of the isolated exact solution can be found in literature. Recently it has been proven that the JT problem possesses  $osp(2, 2)$  symmetry and it is one of the recently discovered quasi-exactly solvable problem [18].

### Dirac Oscillator

The  $(2 + 1)$  dimensional Dirac equation for free particle of mass  $m$  in terms of two component spinors  $\psi$ , can be written as[12, 13]

$$E\psi = \left( \sum_{i=1}^2 c\sigma_i p_i + mc^2\sigma_0 \right) \psi. \quad (12)$$

The momentum operator  $p_i$ , is differential operator  $\mathbf{p} = -i\hbar(\partial_x, \partial_y)$  and the 2D Dirac oscillator can be constructed by changing the momentum  $\mathbf{p} \rightarrow \mathbf{p} - im\omega\sigma_0\mathbf{r}$ . Then the Dirac equation (12) takes the form

$$(E - mc^2\sigma_0)\psi = c[(p_x - ip_y) - im\omega(x - iy)]\sigma_+ + c[(p_x + ip_y) - im\omega(x + iy)]\sigma_-. \quad (13)$$

After some straightforward treatment we obtain the bosonic form of the Dirac oscillator:

$$(E - mc^2\sigma_0)\psi = 2ic\sqrt{m\omega\hbar}[a\sigma_+ + a^+\sigma_-]\psi. \quad (14)$$

An immediate practical consequence of these results is that the Lie algebraic structure of the Hamiltonians can easily be determined.

In addition to those Hamiltonians which we have already bosonised, there exist the Hamiltonians given in terms of bosons and fermions, namely, JC Hamiltonians which can also be treated in the same manner presented in this paper. One of them is known as JC Hamiltonian without rotating wave approximation is given by

$$H_{JC} = \hbar\omega a^+a + \frac{\hbar\omega_0}{2}\sigma_0 + \kappa(\sigma_+ + \sigma_-)(a^+ + a). \quad (15)$$

The other is known as JC Hamiltonian with rotating wave approximation (RWA) can be expressed as

$$H_{JC}^{RWA} = \hbar\omega a^+ a + \frac{\hbar\omega_0}{2} \sigma_0 + \kappa (\sigma_+ a + \sigma_- a^+) \quad (16)$$

which can exactly be solved. When single two-level atom is placed in the common domain of two cavities interacting with two quantized modes, the Hamiltonian of a such system can be obtained from the modification of the JC Hamiltonian and it is given by

$$H_{MJC} = \hbar\omega a^+ a + b^+ b + \hbar\omega_0 \sigma_0 + (\lambda_1 a + \lambda_2 b) \sigma_+ + (\lambda_1 a^+ + \lambda_2 b^+) \sigma_- \quad (17)$$

In a similar manner the two dimensional Hamiltonians can be bosonised and as it will be shown that their algebraic structure can be easily determined. Now, we briefly review construction of the bosonic representations of the Lie algebras  $su(2)$  and  $su(1,1)$ .

### REALIZATIONS OF $su(2)$ AND $su(1,1)$ BY BOSON OPERATORS

It is well known that if the Hamiltonians characterized by single or double boson operators, then the simplest way to find the symmetry algebra of the corresponding Hamiltonian is that to construct the single or double boson realizations of the algebras. In this section we introduce some basic boson realizations of  $su(1,1)$  and  $su(2)$  which we need to solve the Hamiltonians given in previous section.

#### Realizations of $su(2)$

The  $su(2)$  algebra can be constructed by two mode bosons in two dimensions by introducing the following three generators

$$J_+ = a^+ b; \quad J_- = b^+ a; \quad J_0 = \frac{1}{2} (a^+ a - b^+ b) \quad (18)$$

satisfy the commutation relations

$$[J_0, J_{\pm}] = \pm J_{\pm}; \quad [J_+, J_-] = 2J_0. \quad (19)$$

The number operator which commutes with the generators of the  $su(2)$  algebra is given by

$$N = a^+ a + b^+ b. \quad (20)$$

Casimir invariant of  $su(2)$  can be related to  $N$  by

$$C = \frac{1}{4} N(N+2). \quad (21)$$

The eigenvalues of  $C$  are given by

$$\langle C \rangle = j(j+1). \quad (22)$$

It is obvious that the irreducible representations of  $su(2)$  can be characterized by the total boson number  $N = 2j$ . The application of the realization (18) on a set of  $2j+1$  states, leads to the  $(2j+1)$ -dimensional unitary irreducible representation for each  $j = 0, 1/2, 1, \dots$ . If the basis states  $|j, m\rangle$  ( $m = j, j-1, \dots, -j$ ), then the action of the operators on the basis is given by

$$\begin{aligned} J_0 |j, m\rangle &= m |j, m\rangle \\ J_{\pm} |j, m\rangle &= \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle \\ C |j, m\rangle &= j(j+1) |j, m\rangle \end{aligned} \quad (23)$$

Furthermore, the single boson realizations of  $su(2)$  algebra can be constructed by defining three operators

$$J'_+ = \sqrt{2j - N'} a; \quad J'_- = a^+ \sqrt{2j - N'}; \quad J'_0 = j - N'; \quad N' = a^+ a \quad (24)$$

also satisfy the commutation relations of the  $su(2)$  algebra.

### Realizations of $su(1, 1)$ by boson operators

The Lie algebra  $su(1, 1)$  possesses interesting realizations by bosons and more appropriate to solve the many physical problems. Using the set of boson operators (2) we introduce the three operators

$$K_+ = a^+ b^+; \quad K_- = ab; \quad K_0 = \frac{1}{2} (a^+ a + b^+ b + 1). \quad (25)$$

The reader can easily check that the  $K$ 's satisfy the  $su(1, 1)$  commutation relations

$$[K_0, K_{\pm}] = \pm K_{\pm}; \quad [K_+, K_-] = -2K_0. \quad (26)$$

While in the  $su(2)$  case the number operator was the sum,  $N$ , of the boson number operators, in the present case it is the operator

$$M = a^+ a - b^+ b, \quad (27)$$

which is the difference of the number operators. The casimir invariant of the  $su(1, 1)$  is related to  $M$  by

$$C = \frac{1}{4}(1 + M)(1 - M). \quad (28)$$

Therefore if the eigenvalues of the operator  $C$  is  $k(1 - k)$  we find that  $M = (1 - 2k)$ . Consequently, the action of the realization (25) on the states  $|k, n\rangle$ , ( $n = 0, 1, 2, \dots$ ), leads to infinite dimensional unitary irreducible representation, so called positive representation  $D^+(k)$ , corresponds to any  $k = 1/2, 1, 3/2, \dots$ .

$$\begin{aligned} K_0 |k, n\rangle &= (k + n) |k, n\rangle \\ K_+ |k, n\rangle &= \sqrt{(2k + n)(n + 1)} |k, n + 1\rangle \\ K_- |k, n\rangle &= \sqrt{(2k + n - 1)n} |k, n - 1\rangle \\ C |k, n\rangle &= k(1 - k) |k, n\rangle \end{aligned} \quad (29)$$

We will end this section by introducing two different single mode boson realizations of the  $su(1, 1)$  algebra. One of them can be constructed by the operators:

$$L_+ = \frac{1}{2} a^{+2}; \quad L_- = \frac{1}{2} a^2; \quad L_0 = \frac{1}{2} \left( a^+ a + \frac{1}{2} \right); \quad M' = a^+ a; \quad (30)$$

For the single-mode bosonic realization of  $su(1, 1)$  that we require here, the Bargmann index  $k$  is equal to either  $1/4$  or  $3/4$  which split the Hilbert space of the boson space into two independent subspace. The other realization is

$$S_+ = a^+ \sqrt{M' + 2k}; \quad S_- = \sqrt{M' + 2ka}; \quad S_0 = M' + k; \quad k > 0. \quad (31)$$

It will be shown that in particular the realizations (25) and (30) play dominant role on the solution of the Hamiltonian (1).

### METHOD

In our method the bosonised Hamiltonians are connected with  $su(1, 1)$  and  $su(2)$ , as well as  $Sp(4, R)$  Lie algebras. This connection opens the way to an algebraic treatment of a large class of physical Hamiltonians of practical interest. In this section we present a general procedure to solve the Hamiltonian (1). Consider the eigenvalue equation

$$H\psi = E\psi \quad (32)$$

where  $\psi$  is two component wavefunction and  $E$  is eigenvalues of the Hamiltonian  $H$ . Consequently the Hamiltonian (1) can be written as

$$(H_0 - E - \beta) \psi_1 + (\kappa_1 a + \kappa_2 a^+ + \kappa_3 b + \kappa_4 b^+) \psi_2 = 0 \quad (33a)$$

$$(H_0 - E + \beta) \psi_2 + (\gamma_1 a + \gamma_2 a^+ + \gamma_3 b + \gamma_4 b^+) \psi_1 = 0. \quad (33b)$$

Related	Parameters												Symmetry	
Hamiltonian	$\kappa_1$	$\kappa_2$	$\kappa_3$	$\kappa_4$	$\gamma_1$	$\gamma_2$	$\gamma_3$	$\gamma_4$	$\omega_1$	$\omega_2$	$\beta$	Group		
$H_{JT}$	$\kappa'$	0	0	$\kappa'$	0	$\kappa'$	$\kappa'$	0	$\omega$	$\omega$	$\frac{\mu}{2}$	$su(1,1)$	$Eq.(25)$	
$H_{dot}$	$\lambda'$	0	0	$-\lambda'$	0	$\lambda'$	$-\lambda'$	0	$\omega$	$\omega$	$\frac{\mu}{2}gB$	$su(1,1)$	$Eq.(25)$	
$H_{JC}$	$\kappa$	$\kappa$	0	0	$\kappa$	$\kappa$	0	0	$\omega$	0	$\frac{\hbar\omega_0}{2}$	$su(1,1)$	$Eq.(27)$	
$H_{JC}^{RWA}$	$\kappa$	0	0	0	0	$\kappa$	0	0	$\omega$	0	$\frac{\hbar\omega_0}{2}$	$su(1,1)$	$Eq.(27)$	
$H_{MJC}$	$\lambda_1$	0	$\lambda_2$	0	0	$\lambda_1$	0	$\lambda_2$	$\omega$	$\omega$	$\hbar\omega_0$	$su(2)$	$Eq.(18)$	
$H_{Dirac}$	$\kappa''$	0	0	0	0	$\kappa''$	0	0	0	0	$mc^2$	$su(1,1)$	$Eq.(27)$	

TABLE I: List of the Hamiltonians depending on the choices of the parameters of Hamiltonian (1). The corresponding Hamiltonians are illustrated in the first column and their algebras and appropriate realizations are illustrated in the last column. The parameters  $\kappa' = \sqrt{\frac{m\omega}{4\hbar}}\kappa$ ,  $\kappa'' = 2ic\sqrt{m\omega\hbar}$ ,  $\lambda' = \sqrt{\frac{m\omega}{4\hbar}}\lambda_R$ .

These coupled equations may be solved by using various techniques. In here we follow a new strategies. In the first step we eliminate  $\psi_2$  (or  $\psi_1$ ) between the above equation

$$\psi_2 = -(H_0 - E + \beta)^{-1} (\gamma_1 a + \gamma_2 a^+ + \gamma_3 b + \gamma_4 b^+) \psi_1. \quad (34)$$

Substituting (34) in to (33a) we obtain the following equation

$$(H_0 - E - \beta) \psi_1 - (\kappa_1 a + \kappa_2 a^+ + \kappa_3 b + \kappa_4 b^+) (H_0 - E + \beta)^{-1} (\gamma_1 a + \gamma_2 a^+ + \gamma_3 b + \gamma_4 b^+) \psi_1 = 0. \quad (35)$$

The last equation can be solved by performing a suitable realizations of the  $su(1,1)$  and  $su(2)$  algebra. In our formalism, in order to obtain an adequate form of the (35), in the next step, we use the relations

$$\begin{aligned} a (H_0 - E + \beta)^{-1} &= (H_0 - E + \beta - \omega_1)^{-1} a \\ a^+ (H_0 - E + \beta)^{-1} &= (H_0 - E + \beta + \omega_1)^{-1} a^+ \\ b (H_0 - E + \beta)^{-1} &= (H_0 - E + \beta - \omega_2)^{-1} b \\ b^+ (H_0 - E + \beta)^{-1} &= (H_0 - E + \beta + \omega_2)^{-1} b^+ \end{aligned} \quad (36)$$

by setting  $\omega_1 = \omega_2 = \omega$ , we obtain the following general expression

$$\begin{aligned} &(H_0 - E + \beta + \hbar\omega) (H_0 - E + \beta - \hbar\omega) (H_0 - E - \beta) \psi_1 = \\ &\kappa_1 (H_0 - E + \beta + \hbar\omega) (\gamma_1 a^2 + \gamma_2 a a^+ + \gamma_3 a b + \gamma_4 a b^+) \psi_1 + \\ &\kappa_2 (H_0 - E + \beta - \hbar\omega) (\gamma_1 a^+ a + \gamma_2 a^{+2} + \gamma_3 a^+ b + \gamma_4 a^+ b^+) \psi_1 + \\ &\kappa_3 (H_0 - E + \beta + \hbar\omega) (\gamma_1 b a + \gamma_2 b a^+ + \gamma_3 b^2 + \gamma_4 b b^+) \psi_1 + \\ &\kappa_4 (H_0 - E + \beta - \hbar\omega) (\gamma_1 b^+ a + \gamma_2 b^+ a^+ + \gamma_3 b^+ b + \gamma_4 b^{+2}) \psi_1. \end{aligned} \quad (37)$$

It can be checked easily that for some certain values of  $\kappa_i$ ,  $\gamma_i$  and  $\beta$  the equation can be solved in the framework of  $su(1,1)$  or  $su(2)$  Lie algebra techniques and the Hamiltonian can be related to the various physical Hamiltonians of interest. The resulting Hamiltonians are summarized in the ??.

## RESULTS AND DISCUSSIONS

The results of our investigation are that, for the Hamiltonians whose list given in Table 1, are demonstrated in two categories; exactly [21, 23] and quasi-exactly solvable Hamiltonians[24, 26, 27, 29]. It will be shown that, the rotating wave approximated JC Hamiltonian, Dirac oscillator and modified JC Hamiltonian, as they expected, exactly solvable, while JT, JC and quantum dot Hamiltonians are quasi-exactly solvable.

### Exactly solvable Hamiltonians

The JC Hamiltonian with the rotating wave approximation can be obtained by setting  $\kappa_2 = \kappa_3 = \kappa_4 = \gamma_1 = \gamma_3 = \gamma_4 = 0$ ,  $\kappa_1 = \gamma_2 = \kappa$  and  $\beta = \frac{\hbar\omega_0}{2}$ . In this case the formula (37) takes the form:

$$\left(H_0 - E + \frac{\hbar\omega_0}{2} + \hbar\omega\right) \left(H_0 - E + \frac{\hbar\omega_0}{2} - \hbar\omega\right) \left(H_0 - E - \frac{\hbar\omega_0}{2}\right) \psi_1 = \kappa^2 (H_0 - E + \beta + \hbar\omega) aa^+ \psi_1. \quad (38)$$

The algebraic form of the Hamiltonian can be obtained by combining (??) and the  $su(1, 1)$  realization given in (30), yields

$$\left(\hbar\omega M' - E + \frac{\hbar\omega_0}{2} - \hbar\omega\right) \left(\hbar\omega M' - E - \frac{\hbar\omega_0}{2}\right) \psi_1 = \kappa^2 (M' + 1) \psi_1 \quad (39)$$

If the wavefunction  $\psi_1 = \left|\frac{1}{4}, n\right\rangle$  then the action of the (39) on the state leads to the following expression for the eigenvalues of the JC Hamiltonian

$$E = \left(n - \frac{1}{2}\right) \hbar\omega - \frac{\hbar\omega_0}{2} \pm \sqrt{\hbar^2\omega^2 + 4\kappa^2(n+1)} \quad (40)$$

It is obvious that when the coupling constant  $\kappa$  is zero then the result is the eigenvalues of the simple harmonic oscillator.

The other exactly solvable problem is the Dirac oscillator. It can be obtained by setting the parameters  $\omega_1 = \omega_2 = \kappa_2 = \kappa_3 = \kappa_4 = \gamma_1 = \gamma_3 = \gamma_4 = 0$ ,  $\kappa_1 = \gamma_2 = 2ic\sqrt{m\omega\hbar}$  and  $\beta = mc^2$ , yields the following expression:

$$(-E + mc^2) (-E + mc^2) (-E - mc^2) \psi_1 = -4mc^2\hbar\omega (-E + mc^2) aa^+ \psi_1 \quad (41)$$

The algebraic form of the equation, with the realization (30), is given by

$$(-E + mc^2) (-E - mc^2) \psi_1 = -4mc^2\hbar\omega(M' + 1)\psi_1 \quad (42)$$

Then we obtain the following energy values for the Dirac oscillator:

$$E = \pm \sqrt{m^2c^4 - 4\hbar\omega mc^2(n+1)} \quad (43)$$

which is an exact eigenvalues of the Dirac oscillator.

The last exactly solvable problem we consider here is the MJC model. The Hamiltonian can be obtained by choosing the parameters:  $\kappa_2 = \kappa_4 = \gamma_1 = \gamma_3 = 0$ ,  $\kappa_1 = \gamma_2 = \lambda_1$ ,  $\kappa_3 = \gamma_4 = \lambda_2$ , and  $\beta = \hbar\omega_0$  then (37) takes the form:

$$\begin{aligned} & (H_0 - E + \hbar\omega_0 + \hbar\omega) (H_0 - E + \hbar\omega_0 - \hbar\omega) (H_0 - E - \hbar\omega_0) \psi_1 = \\ & \lambda_1 (H_0 - E + \hbar\omega_0 + \hbar\omega) (\lambda_1 aa^+ + \lambda_2 ab^+) \psi_1 + \\ & \lambda_2 (H_0 - E + \hbar\omega_0 + \hbar\omega) (\lambda_1 ba^+ + \lambda_2 bb^+) \psi_1 \end{aligned} \quad (44)$$

We insert (18) and (20), the realization of  $su(2)$ , into (44) we obtain the following expressions

$$\begin{aligned} & (\hbar\omega N - E + \hbar\omega_0 - \hbar\omega) (\hbar\omega N - E - \hbar\omega_0) \psi_1 = \\ & \left( (\lambda_1^2 + \lambda_2^2) \left(1 + \frac{N}{2}\right) + (\lambda_1^2 - \lambda_2^2) J_0 + \lambda_1 \lambda_2 (J_+ + J_-) \right) \psi_1 \end{aligned} \quad (45)$$

The above equation is not diagonal, but it can be diagonalized by similarity transformation, by the operator

$$O = e^{\frac{\alpha}{2}(J_+ - J_-)}. \quad (46)$$

The action of the operator on the generators of  $su(2)$  is given by

$$\begin{aligned} O(J_+ + J_-)O^\dagger &= (J_+ + J_-) \cos \alpha + 2J_0 \sin \alpha \\ OJ_0O^\dagger &= J_0 \cos \alpha - \frac{J_+ + J_-}{2} \sin \alpha \\ ONO^\dagger &= N. \end{aligned} \quad (47)$$

The transformations of the generators of  $su(2)$  by the operator  $O$  are given by

$$\begin{aligned} & (\hbar\omega N - E + \hbar\omega_0 - \hbar\omega) (\hbar\omega N - E - \hbar\omega_0) \psi_1 = \\ & (\lambda_1^2 + \lambda_2^2) \left(1 + \frac{N}{2}\right) \psi_1 + ((\lambda_1^2 - \lambda_2^2) \cos \alpha + 2\lambda_1 \lambda_2 \sin \alpha) J_0 \psi_1 + \\ & \left(\lambda_1 \lambda_2 \cos \alpha - \frac{(\lambda_1^2 - \lambda_2^2)}{2} \sin \alpha\right) (J_+ + J_-) \psi_1 \end{aligned} \quad (48)$$

The Hamiltonian takes the diagonal form when the following condition hold:

$$\alpha = \cos^{-1} \left( \frac{(\lambda_1^2 - \lambda_2^2)}{(\lambda_1^2 + \lambda_2^2)} \right). \quad (49)$$

The final form of the diagonal Hamiltonian under the condition (49) is given by

$$\begin{aligned} & (\hbar\omega N - E + \hbar\omega_0 - \hbar\omega) (\hbar\omega N - E - \hbar\omega_0) \psi_1 = \\ & (\lambda_1^2 + \lambda_2^2) \left(1 + \frac{N}{2}\right) \psi_1 + (\lambda_1^2 + \lambda_2^2) J_0 \psi_1 \end{aligned} \quad (50)$$

It is obvious that when  $\psi_1 = |j, m\rangle$ , which is which is the eigenstate of the operators  $J_0$ , and  $N$ , we obtain the following expression for the eigenvalues of the MJC Hamiltonian:

$$E = (2j - \frac{1}{2})\hbar\omega - \hbar\omega_0 \pm \frac{1}{2} \sqrt{\hbar^2 \omega^2 + 4(\lambda_1^2 + \lambda_2^2)(j + m + 1)}. \quad (51)$$

Without further details, we have solved various physical Hamiltonians in the framework of the method given in section V. Our task is now to show the method given in the previous section can also be applied to obtain QES of the some physical Hamiltonians.

### QES Hamiltonians

In recent years there has been great deal of interest in quantum optical models which reveal new physical phenomena described by Hamiltonians expressed in terms of the boson and fermion operators. It will be shown that such systems can be analyzed using the method presented in this paper. Consequently, their finite number of eigenvalues and associated eigenfunctions can be obtained in the closed form. These systems are said to be QES systems[24, 25, 26, 27, 28, 29, 30].

The JC Hamiltonian without rotating wave approximation is one of the such QES problems that can be associated to the Hamiltonian (1) with the choices of the parameters:  $\omega_2 = \kappa_3 = \kappa_4 = \gamma_3 = \gamma_4 = 0, \kappa_1 = \kappa_2 = \gamma_1 = \gamma_2 = \kappa$ ,  $\beta = \frac{\hbar\omega_0}{2}$  and  $\omega_1 = \omega$ . Thus the general expression (37) takes the form

$$\begin{aligned} & \left(H_0 - E + \frac{\hbar\omega_0}{2} + \hbar\omega\right) \left(H_0 - E + \frac{\hbar\omega_0}{2} - \hbar\omega\right) \left(H_0 - E - \frac{\hbar\omega_0}{2}\right) \psi_1 = \\ & \kappa^2 \left(H_0 - E + \frac{\hbar\omega_0}{2} + \hbar\omega\right) (a^2 + aa^+) \psi_1 + \\ & \kappa^2 \left(H_0 - E + \frac{\hbar\omega_0}{2} - \hbar\omega\right) (a^+ a + a^{+2}) \psi_1. \end{aligned} \quad (52)$$

The JC Hamiltonian can be expressed in terms of the  $su(1, 1)$  generators given in (30):

$$\begin{aligned} & \left(\hbar\omega M' - E + \frac{\hbar\omega_0}{2} + \hbar\omega\right) \left(\hbar\omega M' - E + \frac{\hbar\omega_0}{2} - \hbar\omega\right) \left(\hbar\omega M' - E - \frac{\hbar\omega_0}{2}\right) \psi_1 = \\ & \kappa^2 \left(\hbar\omega M' - E + \frac{\hbar\omega_0}{2} + \hbar\omega\right) (2L_0 + M' + 1) \psi_1 + \\ & \kappa^2 \left(\hbar\omega M' - E + \frac{\hbar\omega_0}{2} - \hbar\omega\right) (2L_+ + M') \psi_1 \end{aligned} \quad (53)$$

Here the Bargmann index  $k$  is  $\frac{1}{4}$  for the even state  $\psi_1 = |\frac{1}{4}, 2n\rangle$  and  $k = \frac{3}{4}$  for odd state  $\psi_1 = |\frac{3}{4}, 2n+1\rangle$ . Under the action of the operators on the even state  $|\frac{1}{4}, 2n\rangle \equiv |2n\rangle$  we obtain the following recurrence relation

$$\begin{aligned} & \left(2\hbar\omega n - E + \frac{\hbar\omega_0}{2} + \hbar\omega\right) \left(2\hbar\omega n - E + \frac{\hbar\omega_0}{2} - \hbar\omega\right) \left(2\hbar\omega n - E - \frac{\hbar\omega_0}{2}\right) |2n\rangle = \\ & \kappa^2 \left(2\hbar\omega n - E + \frac{\hbar\omega_0}{2} + \hbar\omega\right) \left(\sqrt{2n(2n-1)} |2n-2\rangle + (2n+1) |2n\rangle\right) + \\ & \kappa^2 \left(2\hbar\omega n - E + \frac{\hbar\omega_0}{2} - \hbar\omega\right) \left(\sqrt{(2n+1)(2n+2)} |2n+2\rangle + 2n |2n\rangle\right) \end{aligned} \quad (54)$$

The eigenvalues can be obtained from the recurrence relation and its ground state energy is given by

$$n = 0; E = \frac{1}{2} \left(-\hbar\omega \pm \sqrt{4\kappa^2 + \hbar^2(\omega - \omega_0)^2}\right). \quad (55)$$

In order to obtain the remaining eigenvalues, it is enough to perform calculation in (55), but for higher values of  $n$  (i.e.  $n > 5$ ) it requires numerical treatments.

Another physically important and QES Hamiltonian is the Hamiltonian of the quantum a dot including spin orbit coupling obtained from the formula (37), by setting the parameters:  $\kappa_2 = \kappa_3 = \gamma_1 = \gamma_4 = 0$ , we obtain  $\kappa_4 = -\kappa_1 = -\gamma_2 = \gamma_3 = -\sqrt{\frac{m\omega}{4\hbar}}\lambda_R$  and  $\beta = \frac{1}{2}g\mu B$ :

$$\begin{aligned} & \left(H_0 - E + \frac{1}{2}g\mu B + \hbar\omega\right) \left(H_0 - E + \frac{1}{2}g\mu B - \hbar\omega\right) \left(H_0 - E - \frac{1}{2}g\mu B\right) \psi_1 = \\ & \frac{m\omega}{4\hbar}\lambda_R^2 \left(H_0 - E + \frac{1}{2}g\mu B + \hbar\omega\right) (aa^+ + ab) \psi_1 + \\ & \frac{m\omega}{4\hbar}\lambda_R^2 \left(H_0 - E + \frac{1}{2}g\mu B - \hbar\omega\right) (b^+a^+ + b^+b) \psi_1 \end{aligned} \quad (56)$$

The appropriate Lie algebra of this system is  $su(1,1)$  and it can be written in terms of the generators (25):

$$\begin{aligned} & \left(2\hbar\omega K_0 - E + \frac{1}{2}g\mu B\right) \left(2\hbar\omega K_0 - E + \frac{1}{2}g\mu B - 2\hbar\omega\right) \left(2\hbar\omega K_0 - E - \frac{1}{2}g\mu B - \hbar\omega\right) \psi_1 = \\ & \frac{m\omega}{4\hbar}\lambda_R^2 \left(2\hbar\omega K_0 - E + \frac{1}{2}g\mu B\right) \left(K_0 + K_- + \frac{M}{2}\right) \psi_1 + \\ & \frac{m\omega}{4\hbar}\lambda_R^2 \left(2\hbar\omega K_0 - E + \frac{1}{2}g\mu B - 2\hbar\omega\right) \left(K_0 + K_+ - \frac{M+1}{2}\right) \psi_1. \end{aligned} \quad (57)$$

The basis function of the (57) is  $\psi_1 = |k, n\rangle$ , by using the action of the generators on the state, we obtain the three term recurrence relation

$$\begin{aligned} & (2\hbar\omega(k+n) - E_+) (2\hbar\omega(k+n) - E_+ - 2\hbar\omega) (2\hbar\omega(k+n) - E_- - \hbar\omega) |k, n\rangle = \\ & \frac{m\omega}{4\hbar}\lambda_R^2 (2\hbar\omega(k+n) - E_+) \left((n + \frac{1}{2}) |k, n\rangle + \sqrt{(2k+n-1)n} |k, n-1\rangle\right) + \\ & \frac{m\omega}{4\hbar}\lambda_R^2 (2\hbar\omega(k+n) - E_+ - 2\hbar\omega) \left((2k+n-1) |k, n\rangle + \sqrt{(2k+n)(n+1)} |k, n+1\rangle\right). \end{aligned} \quad (58)$$

where  $E_{\pm} = E \mp \frac{1}{2}g\mu B$ . The solution of the recurrence relation for each values of  $k = 0, 1, 2, \dots$  and  $n = 0, 1, 2, \dots, k$  gives an expression for the eigenvalues of the Hamiltonian  $H_{dot}$ .

Our last example is the JT Hamiltonian which can also be treated as the QES problem. In this case the parameters take the values:  $\gamma_1 = \gamma_4 = \kappa_2 = \kappa_3 = 0$ ,  $\gamma_2 = \gamma_3 = \kappa_1 = \kappa_4 = \sqrt{\frac{m\omega}{4\hbar}}\kappa$  and  $\beta = \frac{\mu}{2}$ . Thus the eigenvalue equation takes the form:

$$\begin{aligned} & \left(H_0 - E + \frac{\mu}{2} + \hbar\omega\right) \left(H_0 - E + \frac{\mu}{2} - \hbar\omega\right) \left(H_0 - E - \frac{\mu}{2}\right) \psi_1 = \\ & \frac{m\omega}{4\hbar}\kappa^2 \left(H_0 - E + \frac{\mu}{2} + \hbar\omega\right) (aa^+ + ab) \psi_1 + \\ & \frac{m\omega}{4\hbar}\kappa^2 \left(H_0 - E + \frac{\mu}{2} - \hbar\omega\right) (b^+a^+ + b^+b) \psi_1. \end{aligned} \quad (59)$$

The JT problem can also possess  $su(1,1)$  symmetry and in terms of its generators it can be written as

$$\begin{aligned} & \left(2\hbar\omega K_0 - E + \frac{\mu}{2}\right) \left(2\hbar\omega K_0 - E + \frac{\mu}{2} - 2\hbar\omega\right) \left(2\hbar\omega K_0 - E - \frac{\mu}{2} - \hbar\omega\right) \psi_1 = \\ & \frac{m\omega}{4\hbar} \kappa^2 \left(2\hbar\omega K_0 - E + \frac{\mu}{2}\right) \left(K_- + K_0 + \frac{M+1}{2}\right) \psi_1 + \\ & \frac{m\omega}{4\hbar} \kappa^2 \left(2\hbar\omega K_0 - E + \frac{\mu}{2} - 2\hbar\omega\right) \left(K_+ + K_0 - \frac{M+1}{2}\right) \psi_1. \end{aligned} \quad (60)$$

This algebraic equation with the basis  $\psi_1 = |k, n\rangle$ , leads to the three term recurrence relation:

$$\begin{aligned} & (2\hbar\omega(k+n) - E^+) (2\hbar\omega(k+n) - E^+ - 2\hbar\omega) (2\hbar\omega(k+n) - E^- - \hbar\omega) |k, n\rangle = \\ & \frac{m\omega}{4\hbar} \kappa^2 (2\hbar\omega(k+n) - E^+) \left(\sqrt{(2k+n-1)n} |k, n-1\rangle + (n+1) |k, n\rangle\right) + \\ & \frac{m\omega}{4\hbar} \kappa^2 (2\hbar\omega(k+n) - E^+ - 2\hbar\omega) \left(\sqrt{(2k+n)(n+1)} |k, n+1\rangle + (2k+n-1) |k, n\rangle\right). \end{aligned} \quad (61)$$

where  $E^\pm = E \mp \frac{\mu}{2}$ . The eigenvalues of the JT Hamiltonian can be obtained by solving the recurrence relation (61).

As a consequence we have demonstrated that the solution of the Hamiltonian (1) can be treated within the method presented in this paper for certain values of the parameters. Our approach is relatively simple when compared previous approaches.

## CONCLUSION

Here we have systematically discussed exact and QES of the Hamiltonian (1), within the framework of  $su(2)$  and  $su(1,1)$  Lie algebra. The technique given here can be used in determining the spectrum of the variety of physical systems. We have shown that our formulation leads to the exact or quasi-exact solution of the problems of the various physical systems.

As a further work the technique can be developed such that one can construct  $Sp(4, R)$  algebra which include both  $su(2)$  and  $su(1,1)$  algebras to study the general Hamiltonian (1). The simplest way to construct this algebra is by using boson approach discussed in section III. In addition to the generators  $J_\pm, K_\pm, L_\pm, J_0, K_0, L_0, N, M$ , and  $M'$ , the operators

$$T_+ = \frac{1}{2}b^{+2}; \quad T_- = \frac{1}{2}b; \quad T_0 = \frac{1}{2}b^+b + \frac{1}{4}$$

forms  $Sp(4, R)$  algebra. One can show that the general expression (37) can be expressed in terms of the generators of the  $Sp(4, R)$  algebra:

$$\begin{aligned} & (\hbar\omega N - E + \beta + \hbar\omega) (\hbar\omega N - E + \beta - \hbar\omega) (\hbar\omega N - E - \beta) \psi_1 = \\ & \kappa_1 (\hbar\omega N - E + \beta + \hbar\omega) (\gamma_1 L_- + \gamma_2(1+M) + \gamma_3 K_- + \gamma_4 J_-) \psi_1 + \\ & \kappa_2 (\hbar\omega N - E + \beta - \hbar\omega) (\gamma_1 M + \gamma_2 L_+ + \gamma_3 J_+ + \gamma_4 K_+) \psi_1 + \\ & \kappa_3 (\hbar\omega N - E + \beta + \hbar\omega) (\gamma_1 K_- + \gamma_2 J_+ + \gamma_3 T_- + \gamma_4(1+M''')) \psi_1 + \\ & \kappa_4 (\hbar\omega N - E + \beta - \hbar\omega) (\gamma_1 J_- + \gamma_2 K_+ + \gamma_3 M''' + \gamma_4 T_+) \psi_1. \end{aligned}$$

As we have seen the Hamiltonian can be expressed in terms of the generators the  $Sp(4, R)$  algebra without any constraints on the parameters. Furthermore, we note that the Hamiltonians including higher order interaction terms may also be solved within the method discussed here.

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